

Effects of $f(R)$ Model on the Dynamical Instability of Expansionfree Gravitational Collapse

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Abstract

Dark energy models based on $f(R)$ theory have been extensively studied in literature to realize the late time acceleration. In this paper, we have chosen a viable $f(R)$ model and discussed its effects on the dynamical instability of expansionfree fluid evolution generating a central vacuum cavity. For this purpose, contracted Bianchi identities are obtained for both the usual matter as well as dark source. The term dark source is named to the higher order curvature corrections arising from $f(R)$ gravity. The perturbation scheme is applied and different terms belonging to Newtonian and post Newtonian regimes are identified. It is found that instability range of expansionfree fluid on external boundary as well as on internal vacuum cavity is independent of adiabatic index Γ but depends upon the density profile, pressure anisotropy and $f(R)$ model.

Keywords: $R + \delta R^2$ gravity; Instability; Expansionfree evolution.

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1 Introduction

The "modified gravity" has become a standard terminology for the theories describing gravitational interactions which differ from the most conventional theory of general relativity (GR). In these modified theories, $f(R)$ gravity is able to mimic the standard Λ CDM cosmological evolution and dark energy (DE) problem. Since the laws of gravity gets modified on large distances in $f(R)$ models, this leaves several interesting observational signatures such as modification to the spectra of the galaxy clustering [1], cosmic microwave background [2] and weak lensing [3]. These models have some vacuum solutions with null scalar curvature that allow to recover certain GR solutions. In addition, it has many other applications such as inflation, local gravity constraints, cosmological perturbations and spherically symmetric solutions in weak and strong gravitational backgrounds.

The most important feature of $f(R)$ gravity is to provide the very natural gravitational alternative for DE without adding any matter component. Let us now show that how it can be related to the problem of DE by a straightforward argument. When the Einstein-Hilbert (EH) gravitational action in GR,

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \quad (1.1)$$

is written in the modified form as follows

$$S_{f(R)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R), \quad (1.2)$$

the addition of a non-linear function of the Ricci scalar demonstrates to cause acceleration for a wide variety of $f(R)$ function, e.g., [4]-[13]. Variation of $f(R)$ action with respect to the metric tensor leads to the following fourth order partial differential equations

$$F(R)R_{\alpha\beta} - \frac{1}{2}f(R)g_{\alpha\beta} - \nabla_\alpha \nabla_\beta F(R) + g_{\alpha\beta} \square F(R) = \kappa T_{\alpha\beta}, \quad (\alpha, \beta = 0, 1, 2, 3), \quad (1.3)$$

where $F(R) \equiv df(R)/dR$. Writing this equation in the form of Einstein tensor, it follows that [14]

$$G_{\alpha\beta} = \frac{\kappa}{F} (T_{\alpha\beta}^m + T_{\alpha\beta}^{(D)}), \quad (1.4)$$

where

$$T_{\alpha\beta}^{(D)} = \frac{1}{\kappa} \left[\frac{f(R) - RF(R)}{2} g_{\alpha\beta} + \nabla_\alpha \nabla_\beta F(R) - g_{\alpha\beta} \square F(R) \right]. \quad (1.5)$$

Equation (1.4) shows that effective stress-energy tensor $T_{\alpha\beta}^{(D)}$ plays the role of matter source in the field equations with purely geometrical origin. This approach may provide all the matter ingredients needed to tackle the dark side of the universe. Thus if we restrict "dark source" such that it does not satisfy the usual energy conditions then it can play the role of both dark matter and DE. Consequently, this theory may be used to explain the expansion of the universe and effects of DE in gravitational phenomena. This approach is sometimes convenient to use when we study the DE equation of state [15] as well as the equilibrium description of thermodynamics for the horizon entropy [16]. The trace of Eq.(1.3) determines the dynamics of the scalar field $F(R)$ given by

$$F(R)R - 2f(R) + 3\square F(R) = \kappa T. \quad (1.6)$$

Notice that there exists a de Sitter point which corresponds to a vacuum solution ($T = 0$) at which the Ricci scalar is constant. Since $\square F(R) = 0$ at this point, so we obtain from the above equation

$$F(R)R - 2f(R) = 0. \quad (1.7)$$

It is mentioned here that the model $f(R) = \delta R^2$ satisfies this condition and yields the exact de Sitter solution [17]. In the model $f(R) = R + \delta R^2$, because of the linear term in R , the inflationary expansion ends when δR^2 becomes smaller than the linear term. This is followed by a reheating stage in which the oscillation of R leads to the gravitational particle production. It is also possible to use the de Sitter point for DE.

The gravitational collapse is an important and long standing issue in GR. Recently, it has gained attention in modified theories as well. During gravitational collapse, self-gravitating objects may pass through phases of intense dynamical activities for which quasi-static approximation is not reliable. For instance, the collapse of very massive stars [18], the quick collapse phase yielding neutron star formation [19] and the peculiar stars. The dynamical equations are used to observe the collapsing process while the vanishing expansion scalar condition is developed in connection with the description of

voids. The vanishing of expansion scalar requires that the innermost shell of the fluid should be away from the center, initiating therefrom the formation of the cavity within the matter distribution [20, 21]. This natural appearance of a vacuum cavity suggests that they might be relevant for the modelling of cosmological voids. The Skripkin model is the first example satisfying expansionfree condition [22]. This model corresponds to evolution of spherically symmetric non-dissipating fluid distribution with constant energy density. It is also remarked that the expansionfree condition is a sufficient but not a necessary condition for the appearance of cavities. Cavities described under different kinematical condition are discussed by Herrera et al [23].

For the physically relevant models, the expansionfree evolution requires pressure anisotropy in the fluid distribution and inhomogeneity in the energy density. To assume isotropy of the pressure together with the expansion-free condition, we impose that the energy-density is independent on the time-like coordinate, which severely restricts the models [24]. A stellar model can exist only if it is stable against fluctuations. The problem of dynamical instability is closely related to the structure formation and evolution of self-gravitating objects. Chandrasekhar [25] was the first who worked in this direction. Afterwards, this issue has been investigated by many authors for adiabatic, non-adiabatic, anisotropic and shearing viscous fluids [26]-[30]. We have investigated the problem of DE and gravitational collapse in $f(R)$ gravity [31, 32].

In this paper, we are concerned with spherically symmetric stars having locally anisotropic fluid distribution inside and would investigate how $f(R)$ terms affect the dynamical instability of expansionfree fluid evolution. The format of the paper is as follows. In section 2, basic equations are given. Section 3 is devoted to study the perturbation scheme and a well-known physical $f(R)$ model. In section 4, Newtonian and post Newtonian approximations are taken into account and different terms belonging to these regimes are identified. Also, dynamical equations are investigated under the conditions of vanishing scalar and instability conditions of fluid evolution are discussed. The last section 5 provides the summary of the work.

2 Field Equations and Dynamical Equations

We consider a $3D$ hypersurface $\Sigma^{(e)}$, an external boundary of the collapsing spherically symmetric star, which divides a $4D$ spacetime into two regions

named as interior and exterior spacetimes. The interior spacetime to $\Sigma^{(e)}$ is described by the most general spherically symmetric metric as follows

$$ds_-^2 = A^2(t, r)dt^2 - B^2(t, r)dr^2 - C^2(t, r)(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.1)$$

The exterior spacetime is described by the Schwarzschild metric given by

$$ds_+^2 = \left(1 - \frac{2M}{r}\right) d\nu^2 + 2drd\nu - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.2)$$

where M represents the total mass of the system inside the boundary surface $\Sigma^{(e)}$ and ν is the retarded time. The fluid filling the spherically symmetric star is assumed to be locally anisotropic with inhomogeneous energy density. Thus the energy-momentum tensor for such a fluid is given by

$$T_{\alpha\beta} = (\rho + p_\perp)u_\alpha u_\beta - p_\perp g_{\alpha\beta} + (p_r - p_\perp)\chi_\alpha \chi_\beta, \quad (2.3)$$

where ρ is the energy density, p_\perp the tangential pressure, p_r the radial pressure, u_α the four-velocity of the fluid and χ_α is the unit four-vector along the radial direction. These quantities satisfy the relations

$$u^\alpha u_\alpha = 1, \quad \chi^\alpha \chi_\alpha = -1, \quad \chi^\alpha u_\alpha = 0 \quad (2.4)$$

and are obtained from the following definitions in co-moving coordinates

$$u^\alpha = A^{-1}\delta_0^\alpha, \quad \chi^\alpha = B^{-1}\delta_1^\alpha. \quad (2.5)$$

The expansion scalar, Θ , is given by

$$\Theta = u_{;\alpha}^\alpha = \frac{1}{A} \left(\frac{\dot{B}}{B} + 2\frac{\dot{C}}{C} \right), \quad (2.6)$$

where dot and prime represent derivatives with respect to t and r respectively. For spherically symmetric interior metric, the field equations (1.4) become

$$\left(\frac{2\dot{B}}{B} + \frac{\dot{C}}{C}\right) \frac{\dot{C}}{C} - \left(\frac{A}{B}\right)^2 \left[\frac{2C''}{C} + \left(\frac{C'}{C}\right)^2 - \frac{2B'C'}{BC} - \left(\frac{B}{C}\right)^2 \right] = \frac{\kappa}{F} [\rho A^2 + \frac{A^2}{\kappa} \left\{ \frac{f - RF}{2} + \frac{F''}{B^2} + \left(\frac{2\dot{C}}{C} - \frac{\dot{B}}{B}\right) \frac{\dot{F}}{A^2} + \left(\frac{2C'}{C} - \frac{B'}{B}\right) \frac{F'}{B^2} \right\}] , \quad (2.7)$$

$$-2 \left(\frac{\dot{C}'}{C} - \frac{\dot{C}A'}{CA} - \frac{\dot{B}C'}{BC} \right) = \frac{1}{F} \left(\dot{F}' - \frac{A'}{A} \dot{F} - \frac{\dot{B}}{B} F' \right) , \quad (2.8)$$

$$- \left(\frac{B}{A}\right)^2 \left[\frac{2\ddot{C}}{C} - \left(\frac{2\dot{A}}{A} - \frac{\dot{C}}{C}\right) \frac{\dot{C}}{C} \right] + \left(\frac{2A'}{A} + \frac{C'}{C}\right) \frac{C'}{C} - \left(\frac{B}{C}\right)^2 = \frac{\kappa}{F} [p_r B^2 - \frac{B^2}{\kappa} \left\{ \frac{f - RF}{2} - \frac{\ddot{F}}{A^2} + \left(\frac{\dot{A}}{A} + \frac{2\dot{C}}{C}\right) \frac{\dot{F}}{A^2} + \left(\frac{A'}{A} + \frac{2C'}{C}\right) \frac{F'}{B^2} \right\}] , \quad (2.9)$$

$$- \left(\frac{C}{A}\right)^2 \left[\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{A}}{A} \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{\dot{B}\dot{C}}{BC} \right] + \left(\frac{C}{B}\right)^2 \left[\frac{A''}{A} + \frac{C''}{C} - \frac{A'B'}{AB} + \left(\frac{A'}{A} - \frac{B'}{B}\right) \frac{C'}{C} \right] = \frac{\kappa}{F} \left[p_\perp C^2 - \frac{C^2}{\kappa} \left\{ \frac{f - RF}{2} - \frac{\ddot{F}}{A^2} + \frac{F''}{B^2} + \frac{\dot{F}}{A^2} \times \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \left(\frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C}\right) \frac{F'}{B^2} \right\} \right] . \quad (2.10)$$

The dynamical equations help to study the properties of collapsing process. To formulate these equations, the mass function introduced by Misner and Sharp is defined as follows [33]

$$m(t, r) = \frac{C}{2} (1 + g^{\mu\nu} C_{,\mu} C_{,\nu}) = \frac{C}{2} \left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2} \right) . \quad (2.11)$$

This equation provides the total energy inside a spherical body of radius " C ". From the continuity of the first and second differential forms, the matching of the adiabatic sphere to the Schwarzschild spacetime on the boundary surface, $\Sigma^{(e)}$, yields the following result [21, 34]

$$M \stackrel{\Sigma^{(e)}}{=} m(t, r) \quad (2.12)$$

and

$$2 \left(\frac{\dot{C}'}{C} - \frac{\dot{C}A'}{CA} - \frac{\dot{B}C'}{BC} \right) \stackrel{\Sigma^{(e)}}{=} -\frac{B}{A} \left[\frac{2\ddot{C}}{C} - \left(\frac{2\dot{A}}{A} - \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} \right] + \frac{A}{B} \left[\left(\frac{2A'}{A} + \frac{C'}{C} \right) \frac{C'}{C} - \left(\frac{B}{C} \right)^2 \right]. \quad (2.13)$$

Using the field equations (2.8) and (2.9) in the above equation, we obtain

$$-p_r \stackrel{\Sigma^{(e)}}{=} \frac{T_{11}^{(D)}}{B^2} + \frac{T_{01}^{(D)}}{AB}. \quad (2.14)$$

As the expansion scalar describes the rate of change of small volumes of the fluid. It is interesting to note that expansionfree models define the two hypersurfaces, one separating the fluid distribution externally from the Schwarzschild vacuum solution while the other is boundary between the central Minkowskian cavity and the fluid. Taking $\Sigma^{(i)}$ (i stands for internal) to be the boundary surface of that vacuum cavity and matching it with Minkowski spacetime, we have

$$m(t, r) \stackrel{\Sigma^{(i)}}{=} 0, \quad -p_r \stackrel{\Sigma^{(i)}}{=} \frac{T_{11}^{(D)}}{B^2} + \frac{T_{01}^{(D)}}{AB}. \quad (2.15)$$

The physical applications of such models lie at the core of astrophysical background where the cavity within the fluid distribution is present. It may help to investigate the voids on cosmological scales [35]. Voids are the sponge-like structures and occupying 40%-50% volume of the universe. There exist different sizes of the voids, i.e., mini-voids [36] to super-voids [37]. Observational data shows that voids are neither empty nor spherical. However, for the sake of investigations they are usually described as vacuum spherical cavities surrounding by the fluid.

The proper time and radial derivatives are given by

$$D_T = \frac{1}{A} \frac{\partial}{\partial t}, \quad D_C = \frac{1}{C'} \frac{\partial}{\partial r}, \quad (2.16)$$

where C is the areal radius of the spherical surface. The velocity of the collapsing fluid is defined by the proper time derivative of C , i.e.,

$$U = D_T C = \frac{\dot{C}}{A} \quad (2.17)$$

which is always negative. Using this expression, Eq.(2.11) implies that

$$E \equiv \frac{C'}{B} = \left[1 + U^2 + \frac{2m}{C} \right]^{1/2}. \quad (2.18)$$

The rate of change of mass in Eq.(2.11) with respect to proper time, with the use of Eqs.(2.7)-(2.10), is given by

$$D_T m = \frac{-\kappa}{2F} \left[\left(p_r + \frac{T_{11}^{(D)}}{B^2} \right) U - E \frac{T_{01}^{(D)}}{AB} \right] C^2. \quad (2.19)$$

This represents variation of total energy inside a collapsing surface of radius C . The presence of dark fluid components shows the contribution of DE having large negative pressure. These terms appear with positive sign representing negative effect, hence decrease the rate of change of mass with respect to time. Now, it depends upon the strength of DE terms that they may balance the positive effect of effective radial pressure or overcome on them. Likewise, we have

$$D_C m = \frac{\kappa}{2F} \left[\rho + \frac{T_{00}^{(D)}}{A^2} - \frac{U T_{01}^{(D)}}{E AB} \right] C^2. \quad (2.20)$$

This equation describes how energy density and curvature terms influence the mass between neighboring surfaces of radius C in the fluid distribution. Here the rate would decrease in the consecutive surfaces by the repulsive effect of DE. Integration of Eq.(2.20) with respect to " C " leads to

$$m = \kappa \int_0^C \frac{C^2}{2F} \left[\rho + \frac{T_{00}^{(D)}}{A^2} - \frac{U T_{01}^{(D)}}{E AB} \right] dC. \quad (2.21)$$

The dynamical equations can be obtained from the contracted Bianchi identities. Consider the following two equations

$$\left(T^{\alpha\beta} + T^{\alpha\beta (D)} \right)_{;\beta} u_\alpha = 0, \quad \left(T^{\alpha\beta} + T^{\alpha\beta (D)} \right)_{;\beta} \chi_\alpha = 0 \quad (2.22)$$

which yield respectively

$$\frac{\dot{\rho}}{A} + \frac{\dot{B}}{AB}(\rho + p_r) + \frac{2}{A} \frac{\dot{C}}{C}(\rho + p_\perp) + D_1 = 0, \quad (2.23)$$

$$\frac{p'_r}{B} + (\rho + p_r) \frac{A'}{AB} + 2(p_r - p_\perp) \frac{C'}{BC} + D_2 = 0, \quad (2.24)$$

where D_1 and D_2 are components of dark source given in **Appendix** (5.1, 5.2).

3 The $f(R)$ Model and Perturbation Scheme

Obtaining the correct dynamics of the background cosmological model is not sufficient for any theory to be viable. It is practically impossible to separate different $f(R)$ models without using cosmological perturbations. In this section, we apply perturbation scheme on the field equation and dynamical equations in order to investigate the instability conditions of collapsing fluid evolution. We consider following $f(R)$ model

$$f(R) = R + \delta R^2, \quad (3.1)$$

where δ is any real number. When we take conformal transformation of $f(R)$ action, it is found that the Schwarzschild solution is the only static spherically symmetric solution for the above model [38]. The stability criteria for this model is restricted to $\delta > 0$ which corresponds to $f''(R) > 0$. For $\delta = 0$, GR is recovered in which black holes are stable classically but not quantum mechanically due to Hawking radiations. Since such features also found in $f(R)$ gravity, hence the classical stability condition for the Schwarzschild black hole can be expressed as $f''(R) > 0$ [39].

In our perturbation scheme, we assume that initially all the quantities have only radial dependence, i.e., fluid is in static equilibrium. After that all the quantities and the metric functions have time dependence in their perturbation. This is given by

$$A(t, r) = A_0(r) + \epsilon T(t)a(r), \quad (3.2)$$

$$B(t, r) = B_0(r) + \epsilon T(t)b(r), \quad (3.3)$$

$$C(t, r) = C_0(r) + \epsilon T(t)\bar{c}(r), \quad (3.4)$$

$$\rho(t, r) = \rho_0(r) + \epsilon \bar{\rho}(t, r), \quad (3.5)$$

$$p_r(t, r) = p_{r0}(r) + \epsilon \bar{p}_r(t, r), \quad (3.6)$$

$$p_\perp(t, r) = p_{\perp 0}(r) + \epsilon \bar{p}_\perp(t, r), \quad (3.7)$$

$$m(t, r) = m_0(r) + \epsilon \bar{m}(t, r), \quad (3.8)$$

$$\Theta(t, r) = \epsilon \bar{\Theta}(t, r). \quad (3.9)$$

Also, the Ricci scalar in $f(R)$ model follow the same scheme as

$$R(t, r) = R_0(r) + \epsilon T(t)e(r), \quad (3.10)$$

$$f(R) = R_0(1 + 2\delta R_0) + \epsilon T(t)e(r)(1 + 2\delta R_0), \quad (3.11)$$

$$F(R) = (1 + 2\delta R_0) + 2\epsilon \delta T(t)e(r), \quad (3.12)$$

where $0 < \epsilon \ll 1$. By the freedom allowed in radial coordinates, we choose $C_0(r) = r$ as the Schwarzschild coordinate.

The static configuration of the field equations (2.7)-(2.10) is obtained as

$$\begin{aligned} & \frac{1}{1 + 2\delta R_0} \left[\kappa \rho_0 - \frac{\delta R_0^2}{2} + \frac{2\delta R_0''}{B_0^2} + \frac{2\delta R_0'}{B_0^2} \left(\frac{2}{r} - \frac{B_0'}{B_0} \right) \right] \\ &= \frac{1}{(B_0 r)^2} \left(2r \frac{B_0'}{B_0} + B_0^2 - 1 \right), \end{aligned} \quad (3.13)$$

$$\begin{aligned} & \frac{1}{1 + 2\delta R_0} \left[\kappa p_{r0} + \frac{\delta R_0^2}{2} - \frac{2\delta R_0'}{B_0^2} \left(\frac{A_0'}{A_0} + \frac{2}{r} \right) \right] \\ &= \frac{1}{(B_0 r)^2} \left(2r \frac{A_0'}{A_0} - B_0^2 + 1 \right), \end{aligned} \quad (3.14)$$

$$\begin{aligned} & \frac{1}{1 + 2\delta R_0} \left[\kappa p_{\perp 0} + \frac{\delta R_0^2}{2} - \frac{2\delta R_0''}{B_0^2} - \frac{2\delta R_0'}{B_0^2} \left(\frac{1}{r} + \frac{A_0'}{A_0} - \frac{B_0'}{B_0} \right) \right] \\ &= \frac{1}{B_0^2} \left[\frac{A_0''}{A_0} - \frac{A_0'}{A_0} \frac{B_0'}{B_0} + \frac{1}{r} \left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} \right) \right]. \end{aligned} \quad (3.15)$$

Applying the perturbed quantities in Eqs.(3.2)-(3.12) to the field equations (2.7)-(2.10) along with Eqs.(3.13)-(3.15), the resulting equations (5.3)-(5.6) are given in **Appendix**. In its static configuration, the dynamical equation (2.23) is identically satisfied while (2.24) yields

$$\begin{aligned} & p_{r0}' + (\rho_0 + p_{r0}) \frac{A_0'}{A_0} + \frac{2}{r} (p_{r0} - p_{\perp 0}) + \frac{1}{\kappa} \left[\frac{-3R_0'}{2} - 3\delta R_0 R_0' \right. \\ & \left. + \frac{2\delta R_0'}{B_0^2} \frac{B_0'}{B_0} \left(\frac{A_0'}{A_0} + \frac{2}{r} \right) + \frac{4\delta R_0''}{r B_0^2} - \frac{2\delta R_0'}{B_0^2} \frac{A_0''}{A_0} \right] = 0. \end{aligned} \quad (3.16)$$

The perturbed configurations of Eq.(2.23) leads to

$$\frac{1}{A_0} \left[\dot{\bar{\rho}} + (\rho_0 + p_{r0}) \dot{T} \frac{b}{B_0} + 2(\rho_0 + p_{\perp 0}) \dot{T} \frac{\bar{c}}{r} + D_3 \dot{T} \right] = 0,$$

where D_3 is given in **Appendix**. Integrating this equation with respect to time, we get

$$\bar{\rho} = - \left[(\rho_0 + p_{r0}) \frac{b}{B_0} + 2(\rho_0 + p_{\perp 0}) \frac{\bar{c}}{r} + D_3 \right] T. \quad (3.17)$$

The perturbed part of Eq.(2.24) is obtained as

$$\begin{aligned} & \frac{bT}{B_0^2} \left[p'_{r0} + (\rho_0 + p_{r0}) \frac{A'_0}{A_0} + \frac{2}{r} (p_{r0} - p_{\perp 0}) \right] - (\rho_0 + p_{r0}) \left(\frac{a}{A_0} \right)' \frac{T}{B_0} \\ & + 2(p_{r0} - p_{\perp 0}) \left(\frac{\bar{c}}{r} \right)' \frac{T}{B_0} + \frac{\bar{p}'_r}{B_0} + (\bar{\rho} + \bar{p}_r) \frac{A'_0}{A_0} + \frac{2}{r} (\bar{p}_r - \bar{p}_{\perp}) + D_4 = 0 \end{aligned}$$

where a lengthy expression for D_4 is written in **Appendix**. Perturbation on Eq.(2.11) yields

$$m_0 = \frac{r}{2} \left(1 - \frac{1}{B_0^2} \right), \quad (3.18)$$

$$\bar{m} = -\frac{T}{B_0^2} \left[r \left(\bar{c}' - \frac{b}{B_0} \right) + (1 - B_0^2) \frac{\bar{c}}{2} \right]. \quad (3.19)$$

From the matching condition Eq.(2.14) with Eq.(3.6), it follows that

$$p_{r0} \stackrel{\Sigma^{(e)}}{=} \frac{\delta R_0^2}{2} + \frac{2\delta R'_0}{B_0^2} \left(\frac{A'_0}{A_0} - \frac{2}{r} \right), \quad (3.20)$$

$$\begin{aligned} \bar{p}_r & \stackrel{\Sigma^{(e)}}{=} \frac{2\delta e \ddot{T}}{A_0^2} - \frac{2\delta \dot{T}}{A_0 B_0} \left(e' + e \frac{A'_0}{A_0} - \frac{b}{B_0} R'_0 \right) + 2\delta T \left[-e \left(2\delta R_0 + \frac{R_0}{1 + 2\delta R_0} \right. \right. \\ & + \left. \left. \frac{3R_0^2}{2(1 + 2\delta R_0)} \right) + \frac{R'_0}{B_0^2} \left(\frac{a'}{A_0} + \frac{2A'_0}{A_0 B_0} + \frac{2\bar{c}'}{r} - \frac{2\bar{c}}{r^2} + \frac{4b}{B_0 r} \right) \right. \\ & \left. - \frac{1}{B_0^2} \left(\frac{A'_0}{A_0} - \frac{2}{r} \right) \left(e' - \frac{\delta e R'_0}{1 + 2\delta R_0} \right) \right]. \quad (3.21) \end{aligned}$$

Substituting the above equations in Eq.(5.5), we obtain

$$\alpha(r) \ddot{T}(t) + \beta(r) \dot{T}(t) + \gamma(r) T(t) \stackrel{\Sigma^{(e)}}{=} 0, \quad (3.22)$$

where α , β , and γ are provided in **Appendix**. For the sake of instability region, we assume that all the functions involved in the above equation are such

that α , β and γ remain positive. The corresponding solution of Eq.(3.22) is given by

$$T(t) = -\exp(\omega_{\Sigma(e)}t), \quad \text{where} \quad \omega_{\Sigma(e)} = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \quad (3.23)$$

Here we impose that the system starts collapsing at $t = -\infty$ such that $T(-\infty) = 0$, keeping the system in static position. Afterwards with the increase of t , it goes on collapsing. Using Eqs.(3.9) and (2.6), it follows that

$$\bar{\Theta} = \frac{\dot{T}}{A_0} \left(\frac{b}{B_0} + \frac{2\bar{c}}{r} \right). \quad (3.24)$$

The expansionfree condition implies that

$$\frac{b}{B_0} = -2\frac{\bar{c}}{r}. \quad (3.25)$$

It is interesting to mention that expansionfree fluid evolution causes a blowup of shear scalar which results the appearance of a naked singularity [21, 30]. The dynamical instability of collapsing fluids can be well discussed in term of adiabatic index Γ . We relate $\bar{\rho}$ and \bar{p}_r for the static spherically symmetric configuration by assuming an equation of state of Harrison-Wheeler type as follows [27, 40]

$$\bar{p}_r = \Gamma \frac{p_{r0}}{\rho_0 + p_{r0}} \bar{\rho}. \quad (3.26)$$

Here Γ measures the variation of pressure for a given variation of density. We take it constant throughout the fluid evolution.

4 Newtonian and Post Newtonian Terms and Dynamical Instability

This section provides help to identify the terms belonging to Newtonian (N), post Newtonian (pN) and post post Newtonian (ppN) regimes. This is done by converting relativistic units into c.g.s. units and expanding upto order c^{-4} in the dynamical equation. For the N approximation, we assume

$$\rho_0 \gg p_{r0}, \quad \rho_0 \gg p_{\perp 0}. \quad (4.1)$$

For the metric coefficients expanded upto pN approximation, we take

$$A_0 = 1 - \frac{Gm_0}{c^2}, \quad B_0 = 1 + \frac{Gm_0}{c^2}, \quad (4.2)$$

where G is the gravitational constant and c is the speed of light. From Eq.(3.15), we can write

$$\begin{aligned} \frac{A_0''}{A_0} &= \left(\frac{B_0'}{B_0} - \frac{1}{r} - \frac{2\delta R_0'}{1 + 2\delta R_0} \right) + \frac{B_0^2}{1 + 2\delta R_0} \left[\kappa p_{\perp 0} + \frac{\delta R_0^2}{2} \right] \\ &- \frac{2\delta}{1 + 2\delta R_0} \left[R_0'' + R_0' \left(\frac{1}{r} - \frac{B_0'}{B_0} \right) \right] + \frac{1}{r} \frac{B_0'}{B_0}. \end{aligned} \quad (4.3)$$

Also, from Eq.(3.18), we obtain

$$\frac{B_0'}{B_0} = \frac{-m_0}{r(r - 2m_0)}. \quad (4.4)$$

Using Eqs.(3.18) and (3.14), it follows that

$$\frac{A_0'}{A_0} = \frac{2r^3(\kappa p_{r0} - R_0 - 3\delta R_0^2) + 4\delta r(R_0 - 2R_0'r + 4R_0'm_0) + 4m_0}{4r(r - 2m_0)(1 + 2\delta R_0 + \delta R_0'r)}. \quad (4.5)$$

Substituting Eqs.(4.2), (4.3), (4.5) in (3.16) and doing some algebra, the first dynamical equation in relativistic units is as follows

$$\begin{aligned} p_{r0}' &= \frac{2}{r}(p_{\perp 0} - p_{r0}) - \left[\rho_0 + p_{r0} + (r - 2m_0) \frac{2\delta R_0'}{r} \left(\frac{1}{r} + \frac{2\delta R_0'}{1 + 2\delta R_0} \right) \right] \\ &\times \left[\frac{2r^3(\kappa p_{r0} - R_0 - 3\delta R_0^2) + 4\delta r(R_0 - 2R_0'r + 4R_0'm_0) + 4m_0}{4r(r - 2m_0)(1 + 2\delta R_0 + \delta R_0'r)} \right] \\ &- \frac{2\delta R_0'}{\kappa(1 + 2\delta R_0)} \left[\kappa p_{\perp 0} + \frac{\delta R_0^2}{2} \right] + \frac{2\delta(r - 2m_0)R_0'}{\kappa r^2} \\ &\times \left[\frac{m_0}{r(r - 2m_0)} \left(1 + \frac{2\delta r R_0'}{1 + 2\delta R_0} \right) + 2R_0'' \left(\frac{1}{R_0'} + \frac{\delta r}{1 + 2\delta R_0} \right) \right] \\ &+ \frac{2\delta R_0'}{(1 + 2\delta R_0)} + \frac{3R_0'}{2\kappa}(1 + 2\delta R_0). \end{aligned} \quad (4.6)$$

In view of dimensional analysis, this equation can be written in c.g.s.

units as follows

$$\begin{aligned}
p'_{r0} = & -\frac{G}{c^2} \left[\rho_0 + c^{-2} p_{r0} + \left(\frac{r}{Gc^{-2}} - 2m_0 \right) \frac{2\delta R'_0}{r} \left(\frac{1}{r} + \frac{2\delta R'_0}{1+2\delta R_0} \right) \right] \\
& \times \left[\frac{2r^3(\kappa p_{r0} - R_0 - 3\delta R_0^2) + 4\delta r c^4 G^{-1}(R_0 - 2R'_0 r + 4R'_0 Gc^{-2}m_0) + 4m_0 c^2}{4r(r - 2Gc^{-2}m_0)(1 + 2\delta R_0 + \delta R'_0 r)} \right] \\
& - \frac{2\delta R'_0 c^2}{\kappa(1 + 2\delta R_0)} \left[\frac{\kappa c^2 p_{\perp 0}}{G} + c^{-2} \frac{\delta R_0^2}{2} \right] + \frac{2\delta(r - 2m_0 Gc^{-2})c^2 R'_0}{\kappa r^2} \\
& \times \left[\frac{m_0}{r(r - 2Gc^{-2}m_0)} \left(1 + \frac{2\delta r R'_0}{1 + 2\delta R_0} \right) + \frac{2c^2 R''_0}{G} \left(\frac{1}{R'_0} + \frac{\delta r}{1 + 2\delta R_0} \right) \right] \\
& + \frac{2\delta R'_0 c^2}{G(1 + 2\delta R_0)} \left] + \frac{3R'_0}{2\kappa} (1 + 2\delta R_0) + \frac{2}{r} (p_{\perp 0} - p_{r0}). \tag{4.7}
\end{aligned}$$

Expanding upto c^{-4} order and rearranging lengthy calculations, the resulting expression given in **Appendix**. Here the order of c differentiates the terms in (5.12), (5.13) and (5.14) respectively as follows

$$\text{terms of order } c^0 \text{ correspond to N-approximation,} \tag{4.8}$$

$$\text{terms of order } c^{-2} \text{ correspond to pN-approximation,} \tag{4.9}$$

$$\text{terms of order } c^{-4} \text{ correspond to ppN-approximation.} \tag{4.10}$$

This strategy may help to discuss some physical results in a certain regime by discarding terms belonging to other regimes. Applying the expansionfree condition (3.25) on (3.17), we get

$$\bar{\rho} = \left[2(p_{r0} - p_{\perp 0}) \frac{\bar{c}}{r} + D_5 \right] T, \tag{4.11}$$

where D_5 is given in **Appendix**. This equation shows that the perturbed energy density depends on the static pressure anisotropy and higher order corrections. This fact supports the expansionfree condition that change in energy density depends exclusively on pressure anisotropy. Substituting this value of $\bar{\rho}$ in (3.26), we note that the expression \bar{p}_r and also $\bar{\rho} \frac{A'_0}{A_0}$ in (4.5) are of ppN order approximation. Thus we neglect them in the following calculations in order to discuss the instability conditions upto pN order.

Applying expansionfree condition on the dynamical equation (5.8) along with value of T in (3.23) and following the choice of radial functions

$$a(r) = a_0 r, \quad \bar{c}(r) = c_0 r \quad e(r) = e_0 r, \tag{4.12}$$

after tedious algebra, it follows that

$$\begin{aligned}
& 2c_0 p'_{r0} + (p_{r0} - p_{\perp 0}) \frac{4c_0}{r} + (\rho_0 + p_{r0}) \frac{a_0}{A_0} + \frac{4\delta e_0}{\kappa r^2} + \frac{4\delta e_0}{\kappa r} \frac{B'_0}{B_0} \\
& - \frac{2B_0 c_0}{\kappa} \left(\frac{2r\delta R_0}{B_0} \right)' - \frac{B_0}{\kappa} \left[\frac{2\delta R'_0}{B_0^4} \left(\frac{a_0}{A_0} + 8c_0 \right) \right]_{,1} \\
& + \frac{2c_0 B_0^2}{\kappa} \left(\frac{\delta R_0^3}{2B_0^2} \right)' + \frac{2c_0 B_0^2}{r\kappa} \left(\frac{2\delta R'_0}{B_0^4} \right)' + \frac{2\delta R'_0}{\kappa r B_0^2} \left(\frac{4c_0}{r} + \frac{a_0}{A_0^2} \right) \\
& + \frac{4e_0 R_0}{\kappa} \frac{\delta R_0^2}{2} \frac{B'_0}{B_0} + \frac{4\delta R'_0}{\kappa B_0} e_0 r (1 + 2\delta R_0) + \frac{8c_0 B'_0}{\kappa} \\
& - \frac{2\delta B'_0}{\kappa} \left[\frac{2e_0}{r} + \frac{a_0 B_0 R'_0}{A_0} + \frac{12c_0 R'_0}{r} \right] + \frac{32\delta c_0}{\kappa r B_0^2} \left[\frac{R''_0}{B_0} (1 - B_0) - \frac{R'_0}{r} \right] \\
& - \frac{4\delta e_0}{\kappa r B_0^3} \left[\frac{1}{r} - 4c_0 + \frac{B'_0}{B_0} \right] - \frac{A'_0}{A_0} \left[r a_0 (\rho_0 + p_{r0}) + \frac{B_0}{\kappa} \left\{ \frac{2\delta R'_0}{\kappa} (4c_0 \right. \right. \\
& - a_0 r) \Big\}_{,1} + \frac{2\delta e_0}{\kappa} - \frac{2c_0 B_0^2}{\kappa} \left(\frac{2\delta R'_0}{B_0^4} \right)_{,1} + \frac{2\delta}{\kappa r A_0^2 B_0^2} (a_0 r R'_0 + e_0 + 2R''_0) \\
& + \frac{2\delta B'_0 R'_0}{\kappa} \left(\frac{e_0}{R_0} - 10c_0 - 2c_0 B_0 - \frac{a_0 r}{A_0} \right) - 2c_0 p_{r0} \Big] + \left[\frac{4\delta c_0 R'_0}{\kappa B_0^2} \right. \\
& - \left. \frac{2\delta R'_0}{\kappa B_0^3} (4c_0 - a_0 r) - \frac{2\delta e_0}{\kappa} \right] \left(\frac{A'_0}{A_0} \right)_{,1} - \frac{2\delta \omega_{\Sigma^{(e)}}^2}{\kappa} \left[B_0^2 \left(\frac{e_0 r}{A_0^2 B_0^2} \right)_{,1} \right. \\
& + \left. \frac{1}{A_0^2} \left\{ e_0 + 2c_0 R'_0 + e_0 r (A_0 - 1) \frac{A'_0}{A_0} \right\} \right] + \frac{2\bar{p}_\perp B_0}{r} e^{-\omega_{\Sigma^{(e)}} t} = 0. \quad (4.13)
\end{aligned}$$

Inserting Eqs.(4.2), (4.4) and (4.5) upto pN order (with $c = G = 1$) in the above equation, we obtain

$$\begin{aligned}
& 2c_0 p'_{r0} + (p_{r0} - p_{\perp 0}) \frac{4c_0}{r} + (\rho_0 + p_{r0})(r + m_0) \frac{a_0}{r} + \frac{4\delta e_0}{\kappa r^2} + \frac{4\delta e_0 m_0}{\kappa r^3} \\
& \times (r + 2m_0) - \frac{2c_0}{\kappa r} (r + m_0) [(r + m_0)(2\delta R_0)]_{,1} - \frac{1}{r\kappa} (r + m_0) \\
& \times \left[\frac{2\delta R'_0}{r^2} (r - 4m_0)(a_0(r - m_0) + 8c_0 r) \right]_{,1} + \frac{2c_0}{r\kappa} (r + 2m_0) \left[\frac{R_0}{2r} (r + 2m_0) \right. \\
& \times (2 + 3\delta R_0)]_{,1} + \frac{2c_0}{r^2 \kappa} (r + 2m_0) \left[\frac{2\delta R'_0}{r} (r - 4m_0) \right]_{,1} + \frac{2\delta R'_0}{\kappa r^3} (r - 2m_0)
\end{aligned}$$

$$\begin{aligned}
& \times [4c_0 + a_0(r + 2m_0)] + \frac{2e_0 R_0 m_0}{r^3 \kappa} \delta R_0 (r + 2m_0) - \frac{4\delta R'_0 e_0}{\kappa} \\
& \times (r - m_0) 2\delta R_0 + \frac{8c_0 m_0}{r^4 \kappa} (r + 2m_0)^2 - \frac{2\delta m_0^2}{r^5 \kappa} (r + 2m_0)^2 \\
& \times [2e_0 + \frac{a_0 R'_0}{r} (r^2 - m_0^2) + 12c_0 R'_0] + \frac{32\delta c_0}{\kappa r^4} (r - 2m_0) [R''_0 (r - m_0) m_0 \\
& - R'_0 r] - \frac{4\delta e_0}{\kappa r^5} (r - 3m_0) [r - 4c_0 r^2 + m_0 (r + 2m_0)] \\
& - \left[\frac{2r^3 (\kappa p_{r0} + \delta R_0^2) + 4\delta r (-2R'_0 r + 4R'_0 m_0) + 4m_0}{4r(r - 2m_0)(1 + 2\delta R_0 + \delta R'_0 r)} \right] \\
& \times \left[r a_0 (\rho_0 + p_{r0}) + \frac{1}{r \kappa} (r + m_0) \left\{ \frac{2\delta R'_0}{\kappa} (4c_0 - a_0 r) \right\}_{,1} + \frac{2\delta e_0}{\kappa} - \frac{2c_0}{r \kappa} \right. \\
& \times (r + 2m_0) \left\{ \frac{2\delta R'_0}{r} (r - 2m_0) \right\}_{,1} + \frac{2\delta}{\kappa r^3} (r - 2m_0) (r + 2m_0) + (a_0 r R'_0 \\
& + e_0 + 2R''_0) + \frac{2m_0 \delta R'_0}{r^5 \kappa} (r + 2m_0)^2 \{ -10rc_0 + \frac{re_0}{R_0} - 2c_0(r + m_0) - a_0 \\
& \times (r + m_0) \} - 2c_0 p_{r0}] - \frac{2\delta \omega^2}{r \kappa} \left[(r + 2m_0) e_0 \left(\frac{r - 4m_0^2}{r} \right)_{,1} + (r + 2m_0) \right. \\
& \times (e_0 + 2c_0 R'_0)] + \frac{2\bar{p}_\perp}{r^2} (r + m_0) e^{-\omega_{\Sigma(e)} t} + \frac{2\delta}{r \kappa} [2c_0 R'_0 (r + 2m_0) \\
& - R'_0 (r - 3m_0) (4c_0 - a_0 r) - e_0 r + \frac{2\delta \omega_{\Sigma(e)}^2 e_0}{r \kappa} (r - m_0) (r + 2m_0)] \\
& \times \left[\frac{2r^3 (\kappa p_{r0} + \delta R_0^2) + 4\delta r (-2R'_0 r + 4R'_0 m_0) + 4m_0}{4r(r - 2m_0)(1 + 2\delta R_0 + \delta R'_0 r)} \right]_{,1} = 0. \quad (4.14)
\end{aligned}$$

In general, the instability range depends upon the index Γ as it measures the compressibility of the fluid. However, the above equation is independent of Γ which shows that instability region is totally depends upon the pressure anisotropy, energy density, chosen $f(R)$ model and arbitrary constants. Notice that independence of Γ factor indicates that under expansionfree condition, fluid evolves without being compressed. In this way, the given $f(R)$ model shows the consistency of the physical results with expansionfree condition.

Notice that in the above equation, some terms appearing from ppN approximation. In the following, we are interested in discussing the dynamic

instability of N-regime, so we ignore the terms belonging to pN and ppN approximations in Eq.(4.14) as

$$\begin{aligned}
& 2c_0|p'_{r0}| + (p_{r0} - p_{\perp 0})\frac{4c_0}{r} + (\rho_0 + p_{r0})(r + m_0)\frac{a_0}{r} + \frac{4\delta e_0}{\kappa r^2} \\
& + \frac{4\delta e_0 m_0}{\kappa r^3}(r + 2m_0) - \frac{2c_0}{\kappa r}(r + m_0)[(r + m_0)2\delta R_0]_{,1} \\
& - \frac{2c_0}{r\kappa}(r + 2m_0)\left[\frac{R_0}{2r}(r + 2m_0)\delta R_0\right]_{,1} + \frac{2c_0}{r^2\kappa}(r + 2m_0) \\
& \times \left[\frac{2\delta R'_0}{r}(r - 4m_0)\right]_{,1} + \frac{2\delta R'_0}{\kappa r^3}(r - 2m_0)[4c_0 + a_0(r + 2m_0)] \\
& - \frac{4\delta R'_0 e_0}{\kappa}(r - m_0)(2\delta R_0) + \frac{2\bar{p}_\perp}{r^2}(r + m_0)e^{-\omega_{\Sigma(e)}t} \\
& = \frac{4\delta e_0}{\kappa r^4}(r - 3m_0)(1 - 4c_0 r) + \frac{2\delta\omega_{\Sigma(e)}^2}{r\kappa}(r + 2m_0)(e_0 + 2c_0 R'_0) \\
& - \frac{2e_0 R_0 m_0}{r^3\kappa}3\delta R_0(r + 2m_0). \tag{4.15}
\end{aligned}$$

In order to fulfill the instability of expansionfree fluids, we need to keep all the terms positive in Eq.(4.15). Here, we assume that all the arbitrary constants and dynamical quantities are positive whereas $p'_{r0} < 0$ showing that pressure decreases during collapsing process. In addition, we need to satisfy the following constraints

$$p_{r0} > p_{\perp 0}, \quad \frac{1}{4r} > c_0 > 0, \quad r > 4m_0. \tag{4.16}$$

Thus the system would be unstable in N-approximation as long as the above inequalities are satisfied because other constraints on m are followed by the last inequality in Eq.(4.16).

For instance, if we assume that the scalar curvature is constant, i.e., $R_0(r) = R_c = \text{constant}$ and $e_0 = 0$, then Eq.(4.15) reduces to

$$\begin{aligned}
& 2c_0|p'_{r0}| + (p_{r0} - p_{\perp 0})\frac{4c_0}{r} + \frac{R_0 c_0}{r^2\kappa}(r + 2m_0)(1 + 2m'_0)\delta R_0 \\
& + (r + m_0)\left[(\rho_0 + p_{r0})\frac{a_0}{r} + \frac{4c_0}{\kappa r}(1 + m'_0)\delta R_0 + \frac{2\bar{p}_\perp}{r^2}e^{-\omega_{\Sigma(e)}t}\right] = 0. \tag{4.17}
\end{aligned}$$

Applying constant curvature condition on Eq.(2.21) and inserting in the above equation, we have

$$\begin{aligned}
& 2c_0|p'_{r0}| + (p_{r0} - p_{\perp 0})\frac{4c_0}{r} + \frac{R_c c_0}{r^2 \kappa} \left[r + rR_c + \frac{\kappa}{(1 + 2\delta R_c)} \int_{\Sigma^{(i)}}^r \rho_0 r^2 dr \right] \\
& \times \left[R_c + \frac{\kappa \rho_0 r^2}{1 + 2\delta R_c} \right] (2 + 3\delta R_c) + \left[r + \frac{rR_c}{2} + \frac{\kappa}{2(1 + 2\delta R_c)} \right. \\
& \times \left. \int_{\Sigma^{(i)}}^r \rho_0 r^2 dr \right] \left[(\rho_0 + p_{r0})\frac{a_0}{r} + \frac{2\delta R_c c_0}{\kappa r} \left(R_c + \frac{\kappa \rho_0 r^2}{1 + 2\delta R_0} \right) \right. \\
& \left. + \frac{2\bar{p}_{\perp}}{r^2} e^{-\omega_{\Sigma^{(e)}} t} \right] = 0. \tag{4.18}
\end{aligned}$$

Let us consider an energy density profile of the form $\rho_0 = \lambda r^n$, where λ is a positive constant and $-\infty < n < \infty$. Substituting this value of ρ_0 in Eq.(4.18), it follows for $n \neq -3$

$$\begin{aligned}
& 2c_0|p'_{r0}| + (p_{r0} - p_{\perp 0})\frac{4c_0}{r} + \frac{R_c c_0}{r^2 \kappa} \left[r + rR_c + \frac{\kappa \lambda}{3(1 + 2\delta R_c)} (r^{n+3} - r_{\Sigma^{(i)}}^{n+3}) \right] \\
& \times \left[R_c + \frac{\kappa \lambda r^{n+2}}{1 + 2\delta R_c} \right] (2 + 3\delta R_c) + \left[r + \frac{rR_c}{2} + \frac{\kappa \lambda}{6(1 + 2\delta R_c)} \right. \\
& \times \left. (r^{n+3} - r_{\Sigma^{(i)}}^{n+3}) \right] \left[(\lambda r^n + p_{r0})\frac{a_0}{r} + \frac{2\delta R_c c_0}{\kappa r} \left(R_c + \frac{\kappa \lambda r^{n+2}}{1 + 2\delta R_0} \right) \right. \\
& \left. + \frac{2\bar{p}_{\perp}}{r^2} e^{-\omega_{\Sigma^{(e)}} t} \right] = 0. \tag{4.19}
\end{aligned}$$

Thus instability range depends upon the positivity of $p_{r0} - p_{\perp 0}$ and $r^{n+3} - r_{\Sigma^{(i)}}^{n+3}$, i.e., the system would hold the instability of expansionfree fluid for $p_{r0} > p_{\perp 0}$ and $r^{n+3} > r_{\Sigma^{(i)}}^{n+3}$. Finally, it is remarked that close to the Newtonian regime, expansionfree collapse proceeds without compression as the adiabatic index does not involve in all the calculations.

5 Summary

In this paper, we have studied the problem of gravitational collapse in $f(R)$ theory which is strongly motivated by the observational data collected from supernova. Here the higher order curvature terms are thought to be the origin of DE causing acceleration which are treated as the matter part of the field

equations. We have considered the spherically symmetric collapsing stars made up of locally anisotropic fluid and evolving under the expansionfree condition. The curvature terms appear to affect the passive gravitational mass and rate of collapse. Also, $f(R)$ DE slows down the rate of collapse due to its repulsive effect.

The dynamical equations help to investigate the evolution of gravitational collapse with time and yield the variation of total energy inside a collapsing body with respect to time and adjacent surfaces. We have formulated these equations by using contracted Bianchi identities both for the usual matter and effective energy-momentum tensor independently for a well-known $f(R) = R + \delta R^2$ model. The first dynamical equation is used to identify the terms belonging to Newtonian, post Newtonian and post post Newtonian regimes. We have used the concept of relativistic and c.g.s units. The second dynamical equation is used to discuss the instability range of expansionfree fluid evolution upto pN order.

Perturbation scheme is applied on the field equations and dynamical equations. The study of resulting equations shows that the range of instability is independent of adiabatic index Γ which generally plays central role in the definition of instability range. For example, for a Newtonian perfect fluid, the system is unstable for $\Gamma < 4/3$. In our results, the independence of Γ shows the consistency of expansionfree condition with $f(R)$ gravity because this condition requires that fluid would evolve without compressibility. Moreover, the instability range depends upon the anisotropy of radial pressure, energy density and some constraints which arise for keeping the positivity of the dynamical equation in Newtonian approximation. Assumption of constant scalar curvature implies that the above dependence of instability range also describes the instability of cavity itself with the additional information that $r^{n+3} > r_{\Sigma(i)}^{n+3}$ should be satisfied. It is mentioned here that at pN regime only relativistic effects are taken into account, however, physical behavior of the dynamical equation would be the same.

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Appendix

$$\begin{aligned}
D_1 = & \frac{\dot{\rho}}{A} + \frac{\dot{B}}{AB}(\rho + p_r) + \frac{2}{A} \frac{\dot{C}}{C}(\rho + p_\perp) + \frac{A}{\kappa} \left[\left\{ \frac{1}{A^2 B^2} \left(\dot{F}' - \frac{A'}{A} \dot{F} - \frac{\dot{B}}{B} F' \right) \right\}_{,1} \right. \\
& + \left\{ \frac{f - RF}{2A^2} + \frac{F''}{A^2 B^2} - \frac{\dot{F}}{A^2} \left(\frac{\dot{B}}{B} - \frac{2\dot{C}}{C} \right) - \frac{F'}{B^2} \left(\frac{B'}{B} - \frac{2C'}{C} \right) \right\}_{,0} \\
& + \frac{\dot{A}}{A^3} \left\{ \frac{f - RF}{2A^2} + \frac{F''}{A^2 B^2} - \frac{\dot{F}}{A^2} \left(\frac{\dot{B}}{B} - \frac{2\dot{C}}{C} \right) - \frac{F'}{B^2} \left(\frac{B'}{B} - \frac{2C'}{C} \right) \right\} \\
& + \frac{\dot{B}}{BA^2} \left\{ \frac{F''}{B^2} + \frac{\ddot{F}}{A^2} - \frac{\dot{F}}{A^2} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) - \frac{F'}{B^2} \left(\frac{A'}{A} + \frac{B'}{B} \right) \right\} \\
& + \frac{2\dot{C}}{CA^2} \left\{ \frac{\ddot{F}}{A^2} + \frac{\dot{F}}{A^2} \left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A} \right) - \frac{F'}{B^2} \left(\frac{A'}{A} - \frac{C'}{C} \right) \right\} \\
& + \left. \frac{1}{A^2 B^2} \left(\dot{F}' - \frac{A'}{A} \dot{F} - \frac{\dot{B}}{B} F' \right) \left(\frac{2A'}{A} + \frac{B'}{B} + \frac{C'}{C} \right) \right], \tag{5.1}
\end{aligned}$$

$$\begin{aligned}
D_2 = & \frac{p'_r}{B} + (\rho + p_r) \frac{A'}{AB} + 2(p_r - p_\perp) \frac{C'}{BC} - \frac{B}{\kappa} \left[\left\{ \frac{1}{A^2 B^2} \left(\dot{F}' - \frac{A'}{A} \dot{F} - \frac{\dot{B}}{B} F' \right) \right\}_{,0} \right. \\
& + \left\{ \frac{f - RF}{2B^2} - \frac{\ddot{F}}{A^2} + \frac{\dot{F}}{A^2} \left(\frac{\dot{A}}{A} + \frac{2\dot{C}}{C} \right) + \frac{F'}{B^2} \left(\frac{A'}{A} + \frac{2C'}{C} \right) \right\}_{,1} \\
& + \frac{A'}{AB^2} \left\{ \frac{\ddot{F}}{A^2} + \frac{F''}{B^2} - \frac{\dot{F}}{A^2} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) - \frac{F'}{B^2} \left(\frac{A'}{A} - \frac{B'}{B} \right) \right\} \\
& + \frac{2B'}{B^3} \left\{ \frac{f - RF}{2A^2} + \frac{\ddot{F}}{A^2} + \frac{\dot{F}}{A^2} - \frac{\dot{F}}{A^2} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) - \frac{F'}{B^2} \left(\frac{A'}{A} + \frac{B'}{B} \right) \right\} \\
& + \frac{2\dot{C}}{CA^2} \left\{ \frac{\ddot{F}}{A^2} + \frac{\dot{F}}{A^2} \left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A} \right) - \frac{F'}{B^2} \left(\frac{A'}{A} - \frac{C'}{C} \right) \right\} \\
& + \left. \frac{1}{A^2 B^2} \left(\dot{F}' - \frac{A'}{A} \dot{F} - \frac{\dot{B}}{B} F' \right) \left(\frac{\dot{A}}{A} + \frac{3\dot{B}}{B} + \frac{2\dot{C}}{C} \right) \right]. \tag{5.2}
\end{aligned}$$

Perturbed field equations:

$$\begin{aligned}
& \frac{2T}{B_0^2} \left[\left(\frac{\bar{c}}{r} \right)'' - \frac{1}{r} \left(\frac{b}{B_0} \right)' - \left(\frac{B_0'}{B_0} - \frac{3}{r} \right) \left(\frac{\bar{c}}{r} \right)' - \left(\frac{b}{B_0} - \frac{\bar{c}}{r} \right) \left(\frac{B_0}{r} \right)^2 \right] \\
&= \frac{2Tb}{(1+2\delta R_0)B_0} \left[\kappa\rho_0 - \frac{\delta R_0^2}{2} + \frac{2\delta R_0''}{B_0^2} \right] + \frac{\kappa\bar{\rho}}{1+2\delta R_0} \\
&+ \frac{2T}{B_0^2} \frac{2\delta R_0'}{(1+\delta R_0)} \left[\left(\frac{\bar{c}}{r} \right)' - \frac{1}{2} \left(\frac{b}{B_0} \right)' \right], \tag{5.3}
\end{aligned}$$

$$\left(\frac{\bar{c}}{r} \right)' - \frac{b}{B_0 r} - \frac{\bar{c}A_0'}{rA_0} = \frac{\delta}{1+2\delta R_0} \left[-e' + e \frac{A_0'}{A_0} + \frac{bR_0'}{B_0} \right], \tag{5.4}$$

$$\begin{aligned}
& - \frac{2\ddot{T}\bar{c}}{A_0^2 r} + \frac{2T}{rB_0^2} \left[\left(\frac{a}{A_0} \right)' + \left(r \frac{A_0'}{A_0} + 1 \right) \left(\frac{\bar{c}}{r} \right)' - \frac{B_0^2}{r} \left(\frac{b}{B_0} - \frac{\bar{c}}{r} \right) \right] \\
&= \frac{2Tb}{(1+2\delta R_0)B_0} \left(\kappa p_{r0} + \frac{\delta R_0^2}{2} \right) + \frac{\kappa\bar{p}_r}{1+2\delta R_0} + \frac{\ddot{T}}{A_0^2} \frac{2\delta e}{(1+2\delta R_0)} \\
&+ \frac{2T}{rB_0^2} \frac{2\delta R_0'}{(1+\delta R_0)} \left[\left(\frac{a}{A_0} \right)' + 2 \left(\frac{\bar{c}}{r} \right)' \right], \tag{5.5}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\ddot{T}}{A_0^2} \left[\frac{b}{B_0} + \frac{\bar{c}}{r} \right] + \frac{T}{B_0^2} \left[\left(\frac{a}{A_0} \right)'' + \left(\frac{\bar{c}}{r} \right)'' + \left(\frac{2A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r} \right) \left(\frac{a}{A_0} \right)' \right. \\
& - \left. \left(\frac{A_0'}{A_0} + \frac{1}{r} \right) \left(\frac{b}{B_0} \right)' + \left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{2}{r} \right) \left(\frac{\bar{c}}{r} \right)' \right] = \frac{\ddot{T}}{A_0^2} \frac{2\delta e}{1+2\delta R_0} \\
&+ \frac{\kappa\bar{p}_\perp}{1+2\delta R_0} - \frac{T}{B_0^2} \frac{2\delta R_0'}{(1+2\delta R_0)} \left[\left(\frac{a}{A_0} \right)' - \left(\frac{b}{B_0} \right)' + \left(\frac{\bar{c}}{r} \right)' \right] \\
&- \frac{2T}{B_0(1+2\delta R_0)} \left(\kappa p_{\perp 0} + \frac{\delta R_0^2}{2} \right). \tag{5.6}
\end{aligned}$$

$$\begin{aligned}
D_3 = & \frac{1}{A_0\kappa} \left[\frac{3a}{A_0} \frac{\delta R_0^2}{2} - 2\delta e \left(2\delta R_0 + \frac{A_0'}{r^2 A_0 B_0^2} \right) - \frac{2\delta}{\kappa B_0^2} \left(e' + e \frac{A_0'}{A_0} \right) \left(\frac{A_0'}{A_0} - \frac{1}{r} \right) \right. \\
& + \frac{2\delta R_0''}{B_0^2} \left(\frac{3a}{A_0} - \frac{2b}{B_0} \right) + \frac{2\delta R_0'}{B_0^2} \left\{ \frac{2\bar{c}'}{r} + \frac{a}{A_0} \frac{B_0'}{B_0} - \frac{b'}{B_0} + \frac{2b}{B_0} \frac{B_0'}{B_0} - \left(\frac{b}{B_0} \right)' \right. \\
& \left. \left. - \frac{b}{B_0} + \frac{2a}{rA_0} + \frac{4\bar{c}}{r} \left(\frac{1}{r} - \frac{A_0'}{A_0} \right) \right\} \right]. \tag{5.7}
\end{aligned}$$

$$\begin{aligned}
D_4 = & -\frac{B_0 \ddot{T}}{\kappa} \left(\frac{2\delta e}{A_0^2 B_0^2} \right)' - \frac{2\delta \ddot{T}}{\kappa B_0 A_0^2} \left(e' - \frac{A_0' e}{A_0} - \frac{R_0' b}{B_0} \right) - \frac{T}{\kappa} \left(\frac{2\delta R_0 e}{B_0^2} \right)' \\
& + \frac{T}{\kappa} \left[\frac{2\delta R_0'}{B_0^4} \left\{ \left(\frac{a}{A_0} \right)' - \left(\frac{\bar{c}}{r} \right)' - \frac{2b}{B_0} \frac{A_0'}{A_0} - \frac{4b}{r B_0} \right\} + \frac{2\delta e'}{B_0^2} \left(\frac{A_0'}{A_0} + \frac{2}{r} \right) \right]_{,1} \\
& + \frac{Tb}{\kappa^2} \left[\frac{2\delta R_0^2}{2B_0^2} + \frac{2\delta R_0'}{B_0^4} \left(\frac{A_0'}{A_0} + \frac{1}{r} \right) \right]_{,1} - \frac{2\delta e \ddot{T}}{A_0 B_0 \kappa} \frac{A_0'}{A_0} \\
& - \frac{Tb}{A_0 B_0^2 \kappa} \left[\frac{2\delta R_0'' A_0'}{B_0^2} - \frac{2\delta R_0'}{r A_0 B_0^3} \left(\frac{A_0'}{A_0} + \frac{2}{r} \right) \right] + \frac{T}{B_0^2 \kappa} \left[\frac{\bar{c}'' A_0'}{A_0 B_0} - \frac{4\delta R_0''}{B_0} \frac{b}{B_0} \frac{A_0'}{A_0} \right. \\
& + \frac{2\delta R_0''}{A_0 B_0} \left(a' - a \frac{A_0'}{A_0} - 2 \frac{A_0' b}{B_0} \right) - \frac{1}{A_0^2 B_0} \left(2\delta R_0' a - (2\delta R_0' a - e') \frac{A_0'}{A_0} \right. \\
& - \left. \frac{4\delta R_0' b A_0'}{A_0 B_0} \right) - \frac{4\delta R_0'}{r B_0} \left(\bar{c}' - \frac{\bar{c}}{r} \right) - \delta R_0^2 \left(b' - \frac{3b B_0'}{B_0} \right) \\
& + 4\delta B_0' e (1 + 2\delta R_0) + 2\delta e' B_0 B_0' \left(\frac{A_0'}{A_0} + \frac{2}{r} \right) + \frac{2B_0}{r} \left(b' - \frac{3b B_0'}{B_0} \right) \\
& + \frac{4\delta}{r B_0^2} \left(R_0'' - \frac{R_0'}{r} \right) \left(\bar{c}' - \frac{\bar{c}}{r} - \frac{4b}{B_0} \right) - \frac{8\delta R_0'}{r^2 B_0^2} \left(\bar{c}' - \frac{\bar{c}}{r} - \frac{4b}{B_0} \right) + \frac{4\delta e''}{r B_0^2} \\
& + 2\delta R_0' B_0' B_0 \left\{ \frac{A_0'}{A_0} \left(b - a - \frac{5b}{B_0} \right) + \frac{a' B_0}{A_0} + \frac{2\bar{c}'}{r} - \frac{2\bar{c}}{r^3} - \frac{4b}{B_0 r} \right\} \\
& - \frac{4\delta}{r B_0^2} \left(\frac{e' B_0'}{B_0} + \frac{R_0' b'}{B_0} - \frac{R_0' b}{B_0 r} \right) + \frac{4\delta e'}{r^2 B_0^2} \Big]. \tag{5.8}
\end{aligned}$$

$$\alpha(r) \stackrel{\Sigma^{(e)}}{=} \frac{1}{A_0^2} \left(\frac{2\bar{c}}{r} + \frac{\kappa e + e}{1 + 2\delta R_0} \right), \tag{5.9}$$

$$\beta(r) \stackrel{\Sigma^{(e)}}{=} \frac{2\delta \kappa}{A_0 B_0 (1 + 2\delta R_0)} \left(e' + \frac{e A_0'}{A_0} - \frac{b R_0'}{B_0} \right), \tag{5.10}$$

$$\begin{aligned}
\gamma(r) \stackrel{\Sigma^{(e)}}{=} & \frac{\kappa}{B_0^2 (1 + 2\delta R_0)} \left[2\delta e B_0^2 \left\{ -2\delta R_0 + \frac{1}{(1 + 2\delta R_0)} \frac{\delta R_0^2}{2} \right\} \right. \\
& + 2\delta \left(\frac{A_0'}{A_0} - \frac{2}{r} \right) \left(e' - \frac{2\delta e R_0'}{1 + 2\delta R_0} \right) + \frac{2\delta R_0'}{\kappa} \left(\frac{a'}{A_0} + \frac{2A_0'}{A_0 B_0} + \frac{2\bar{c}'}{r} \right. \\
& - \left. \frac{2\bar{c}}{r^2} + \frac{4b}{B_0 r} \right) + \frac{4\delta R_0}{\kappa} \left\{ \left(\frac{a}{A_0} \right)' + \left(r \frac{A_0'}{A_0} + 1 \right) \left(\frac{\bar{c}}{r} \right)' - \frac{B_0^2}{r} \left(\frac{b}{B_0} - \frac{\bar{c}}{r} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + 2bB_0 \left\{ \left(1 + \frac{1}{\kappa} \right) \frac{\delta R_0^2}{2} - \frac{2\delta R'_0}{B_0^2} \left(\frac{A'_0}{A_0} - \frac{2}{r} \right) \right\} \\
& + \frac{4\delta R'_0}{\kappa} \left\{ \left(\frac{a}{A_0} \right)' + 2 \left(\frac{\bar{c}}{r} \right)' \right\} \Bigg]. \tag{5.11}
\end{aligned}$$

$$\begin{aligned}
p'_{r0} &= \frac{2}{r}(p_{\perp 0} - p_{r0}) + \frac{3R'_0}{2\kappa}(1 + 2\delta R_0) - \frac{\delta^2 R'_0 R_0^2}{\kappa}(1 - 2\delta R_0) \\
& - \frac{4\delta R'_0 m_0^2 G}{\kappa r^4}(1 + 2\delta R'_0 r(1 - 2\delta R_0)) + \frac{4\delta^2 R'_0 R_0^2 G}{r\kappa G}(1 - 2\delta R_0) \\
& + \frac{4\delta R''_0}{r\kappa G}(1 + \delta R'_0 r(1 - 2\delta R_0)) - \delta R'_0(\kappa p_{r0} - R_0 - 3\delta R_0^2)(1 + 2\delta R'_0 r \\
& \times (1 - 2\delta R_0)) - \frac{4\delta R'_0 G m_0^2}{r^4}(1 - 2\delta R_0 - \delta R'_0 r)(2G\delta R_0 + 1)(1 + 2r\delta R'_0 \\
& \times (1 + 2\delta R'_0)) \tag{5.12}
\end{aligned}$$

$$\begin{aligned}
& - \frac{G}{c^2 r^3} \left[\left\{ \frac{2\delta R'_0 m_0}{\kappa G} + \frac{8G\delta R'_0 m_0^3}{\kappa r^2} \right\} (1 + 4\delta^2 R'_0 R_0 r) \right. \\
& + \left. \left\{ \frac{r^3}{2} \left(\rho_0 r - \frac{2\delta R'_0 G m_0}{r} + 8\delta^3 R_0'^2 R_0 m_0 \right) (\kappa p_{r0} + \delta R_0^2) \right. \right. \\
& - 2m_0 \delta R'_0 r^2 (\kappa p_{r0} + \delta R_0^2) (1 + 2r\delta^2 R'_0 R_0) \\
& + \frac{2Gm_0^2}{r^2} (1 + GR_0 \delta) (\rho_0 r^2 - 2\delta R'_0 m_0 + 4m_0 \delta^2 R_0'^2 (1 - 2\delta R_0)) \\
& + \left. \left. \frac{8G\delta R'_0 m_0^2}{r} (1 + 4\delta m_0 R'_0) (1 + 2\delta r R'_0) \right\} (1 - 2\delta R_0 - \delta R'_0 r) \right] \tag{5.13}
\end{aligned}$$

$$\begin{aligned}
& - \frac{G}{c^4 r^4} \left[\frac{16\delta R'_0 m_0^4}{\kappa r^2} (1 + 2\delta^2 R'_0 r R_0) + \{ r^5 p_{r0} (\kappa p_{r0} - R_0 - 3\delta R_0^2) \right. \\
& + r^3 G m_0 (\kappa p_{r0} - R_0 - 3\delta R_0^2) \left(G \rho_0 r - \frac{2\delta R'_0 m_0}{r} + 4\delta^3 R_0'^2 R_0 m_0 \right) \\
& + 4Gm_0^2 \delta R'_0 r^2 (\kappa p_{r0} + \delta R_0^2) (1 + 2r\delta^2 R'_0 R_0) + 2rGm_0^2 p_{r0} (1 + 2GR_0 \delta) \\
& + \frac{8\delta G R'_0 m_0^3}{r} (1 + 4r\delta^2 R'_0) (1 + 2r\delta^2 R'_0 R_0) + 4G^2 m_0^3 (1 + 4r\delta R'_0) \\
& \times \left. \left(\rho_0 - \frac{2\delta R'_0 m_0}{r^2} + \frac{4\delta^3 R'_0 R_0^2 m_0}{r} \right) \right\} (1 - 2\delta R_0 - \delta R'_0 r) \Bigg]. \tag{5.14}
\end{aligned}$$

$$\begin{aligned}
D_5 = & \frac{3a}{\kappa A_0} \frac{\delta R_0^2}{2} - \frac{4\delta e''}{\kappa B_0^2} + \frac{2\delta e}{\kappa} \left(2\delta R_0 + \frac{A'_0}{r^2 A_0 B_0^2} \right) + \frac{2\delta}{\kappa B_0^2} \left(e' + e \frac{A'_0}{A_0} \right) \\
& \times \left(\frac{A'_0}{A_0} - \frac{1}{r} \right) - \frac{2\delta R_0''}{B_0^2} \left(\frac{3a}{A_0} + \frac{4\bar{c}}{r} \right) - \frac{2\delta R'_0}{\kappa B_0^2} \left\{ \frac{2B_0 \bar{c}'}{r} + \frac{a}{A_0} \frac{B'_0}{B_0} \right. \\
& \left. + \frac{3\bar{c}'}{r} + \frac{3\bar{c}}{r^2} + \frac{4\bar{c}}{r} \left(\frac{B'_0}{B_0} - \frac{A'_0}{A_0} \right) \right\}. \tag{5.15}
\end{aligned}$$

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